An elliptic semilinear equation with source term and boundary measure data: the supercritical case

Marie-Françoise Bidaut-Véron, Giang Hoang, Quoc-Hung Nguyen, Laurent Véron

An elliptic semilinear equation with source term and boundary measure data: the supercritical case

Marie-Françoise Bidaut-Véron[*]
Giang Hoang[†]
Quoc-Hung Nguyen[‡]
Laurent Véron[§]

Laboratoire de Mathématiques et Physique Théorique,
Université François Rabelais, Tours, FRANCE

Abstract

We give new criteria for the existence of weak solutions to an equation with a super linear source term

$$-\Delta u = u^q \ \text{ in } \Omega, \quad u = \sigma \ \text{ on } \ \partial\Omega$$

where Ω is a either a bounded smooth domain or $\mathbb{R}^N_+$, $q > 1$ and $\sigma \in \mathfrak{M}^+(\partial\Omega)$ is a nonnegative Radon measure on $\partial\Omega$. One of the criteria we obtain is expressed in terms of some Bessel capacities on $\partial\Omega$. We also give a sufficient condition for the existence of weak solutions to equation with source mixed terms.

$$-\Delta u = |u|^{q_1-1}u|\nabla u|^{q_2} \ \text{ in } \Omega, \quad u = \sigma \ \text{ on } \ \partial\Omega$$

where $q_1, q_2 \geq 0, q_1 + q_2 > 1, q_2 < 2$, $\sigma \in \mathfrak{M}(\partial\Omega)$ is a Radon measure on $\partial\Omega$.

1 Introduction and main results

Let Ω be a bounded smooth domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N_+ := \mathbb{R}^{N-1} \times (0, \infty)$, $N \geq 3$, and $g : \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ be a continuous function. In this paper, we study the solvability of the problem

$$\begin{array}{rll} -\Delta u = g(u, \nabla u) & & \text{in } \Omega, \\ u = \sigma & & \text{on } \ \partial\Omega, \end{array} \tag{1.1}$$

where $\sigma \in \mathfrak{M}(\partial\Omega)$ is a Radon measure on $\partial\Omega$. All solutions are understood in the usual very weak sense, which means that $u \in L^1(\Omega)$, $g(u, \nabla u) \in L^1_\rho(\Omega)$, where $\rho(x)$ is the distance from x to $\partial\Omega$ when Ω is bounded, or $u \in L^1(\mathbb{R}^N_+ \cap B)$, $g(u, \nabla u) \in L^1_\rho(\mathbb{R}^N_+ \cap B)$ for any ball B if $\Omega = \mathbb{R}^N_+$, and

$$\int_\Omega u(-\Delta\xi)dx = \int_\Omega g(u, \nabla u)\xi dx - \int_{\partial\Omega} \frac{\partial\xi}{\partial n} d\sigma \tag{1.2}$$

for any $\xi \in C^2(\overline{\Omega}) \cap C_c(\mathbb{R}^N)$ with $\xi = 0$ in Ω^c, where $\rho(x) = \text{dist}(x, \partial\Omega)$, n is the outward unit vector on $\partial\Omega$. It is well-known that such a solution u satisfies

$$u = \mathbf{G}[g(u, \nabla u)] + \mathbf{P}[\sigma] \ \text{ a. e. in } \Omega,$$

[*]E-mail address: veronmf@univ-tours.fr
[†]E-mail address: hgiangbk@gmail.com
[‡]E-mail address: quoc-hung.nguyen@epfl.ch
[§]E-mail address: Laurent.Veron@lmpt.univ-tours.fr

where $\mathbf{G}[.], \mathbf{P}[.]$, respectively the Green and the Poisson potentials associated to $-\Delta$ in Ω, are defined from the Green and the Poisson kernels by

$$\mathbf{P}[\sigma](y) = \int_{\partial\Omega} \mathrm{P}(y,z)d\sigma(z), \quad \mathbf{G}[g(u,\nabla u)](y) = \int_{\Omega} \mathrm{G}(y,x)g(u,\nabla u)(x)dx,$$

see [16].

Our main goal is to establish necessary and sufficient conditions for the existence of weak solutions of (1.1) with boundary measure data, together with sharp pointwise estimates of the solutions. In the sequel we study two cases for the problem (1.1):

1- The pure power case

$$\begin{aligned} -\Delta u &= |u|^{q-1}u \qquad && \text{in } \Omega, \\ u &= \sigma && \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

with $u \geq 0$, $q > 1$ and $\sigma \geq 0$.

2- The mixed gradient-power case

$$\begin{aligned} -\Delta u &= |\nabla u|^{q_2}|u|^{q_1-1}u \qquad && \text{in } \Omega, \\ u &= \sigma && \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

with $q_1, q_2 > 0$, $q_1 + q_2 > 1$ and $q_2 < 2$.

The problem (1.3) has been first studied by Bidaut-Véron and Vivier [2] in the subcritical case $1 < q < \frac{N+1}{N-1}$ with Ω bounded. They proved that (1.3) admits a nonnegative solution provided $\sigma(\partial\Omega)$ is small enough. They also proved that for any $\sigma \in \mathfrak{M}_b^+(\partial\Omega)$ there holds

$$\mathbf{G}[(\mathbf{P}[\sigma])^q] \leq c\sigma(\partial\Omega)\mathbf{P}[\sigma] \tag{1.5}$$

for some $c = c(N, p, q) > 0$. Then Bidaut-Véron and Yarur [3] considered again the problem (1.3) in a bounded domain in a more general situation since they allowed both interior and boundary measure data, giving a complete description of the solutions in the subcritical case, and sufficient conditions for existence in the supercritical case. In particular they showed that the problem (1.3) has a solution if and only if

$$\mathbf{G}[(\mathbf{P}[\sigma])^q] \leq c\mathbf{P}[\sigma] \tag{1.6}$$

for some $c = c(N, q, \Omega) > 0$, see [3, Th 3.12-3.13, Remark 3.12].

The absorption case, i.e. $g(u, \nabla u) = -|u|^{q-1}u$ has been studied by Gmira and Véron [9] in the subcritical case (again $1 < q < \frac{N+1}{N-1}$) and by Marcus and Véron in the supercritical case [13], [15], [16]. The case $g(u, \nabla u) = -|\nabla u|^q$ was studied by Nguyen Phuoc and Véron [17] and extended recently to the case $g(u, \nabla u) = -|\nabla u|^{q_2}|u|^{q_1-1}u$ by Marcus and Nguyen Phuoc [11]. To our knowledge, the problem (1.4) has not yet been studied.

To state our results, let us introduce some notations. We write $A \lesssim (\gtrsim) B$ if $A \leq (\geq) CB$ for some C depending on some structural constants, $A \asymp B$ if $A \lesssim B \lesssim A$. Various capacities will be used throughout the paper. Among them are the Riesz and Bessel capacities in $\mathbb{R}^{N-1}$ defined respectively by

$$\mathrm{Cap}_{I_\gamma,s}(O) = \inf\left\{\int_{\mathbb{R}^{N-1}} f^s dy : f \geq 0, I_\gamma * f \geq \chi_O\right\},$$

$$\mathrm{Cap}_{G_\gamma,s}(O) = \inf\left\{\int_{\mathbb{R}^{N-1}} f^s dy : f \geq 0, G_\gamma * f \geq \chi_O\right\},$$

for any Borel set $O \subset \mathbb{R}^{N-1}$, where $s > 1$, I_γ, G_γ are the Riesz and the Bessel kernels in $\mathbb{R}^{N-1}$ with order $\gamma \in (0, N-1)$. We remark that

$$\mathrm{Cap}_{G_\gamma,s}(O) \geq \mathrm{Cap}_{I_\gamma,s}(O) \geq C|O|^{1-\frac{\gamma s}{N-1}} \tag{1.7}$$

for any Borel set $O \subset \mathbb{R}^{N-1}$ where $\gamma s < N-1$ and C is a positive constant. When we consider equations in a bounded smooth domain Ω in $\mathbb{R}^N$ we use a specific capacity that we define as follows: there exist open sets $O_1, ..., O_m$ in $\mathbb{R}^N$, diffeomorphisms $T_i : O_i \mapsto B_1(0)$ and compact sets $K_1, ..., K_m$ in $\partial\Omega$ such that

a. $K_i \subset O_i$, $\partial\Omega \subset \bigcup_{i=1}^{m} K_i$.

b. $T_i(O_i \cap \partial\Omega) = B_1(0) \cap \{x_N = 0\}$, $T_i(O_i \cap \Omega) = B_1(0) \cap \{x_N > 0\}$.

c. For any $x \in O_i \cap \Omega$, $\exists y \in O_i \cap \partial\Omega$, $\rho(x) = |x - y|$.

Clearly, $\rho(T_i^{-1}(z)) \asymp |z_N|$ for any $z = (z', z_N) \in B_1(0) \cap \{x_N > 0\}$ and $|\mathbf{J}_{T_i}(x)| \asymp 1$ for any $x \in O_i \cap \Omega$, here $\mathbf{J}_{T_i}$ is the Jacobian matrix of T_i.

Definition 1.1 *Let $\gamma \in (0, N-1), s > 1$. We define the $\mathrm{Cap}^{\partial\Omega}_{\gamma,s}$-capacity of a compact set $E \subset \partial\Omega$ by*

$$\mathrm{Cap}^{\partial\Omega}_{\gamma,s}(E) = \sum_{i=1}^{m} \mathrm{Cap}_{G_\gamma,s}(\tilde{T}_i(E \cap K_i)),$$

where $T_i(E \cap K_i) = \tilde{T}_i(E \cap K_i) \times \{x_N = 0\}$.

Notice that, if $\gamma s > N-1$ then there exists $C = C(N, \gamma, s, \Omega) > 0$ such that

$$\mathrm{Cap}^{\partial\Omega}_{\gamma,s}(\{x\}) \geq C \tag{1.8}$$

for all $x \in \partial\Omega$. Also the definition does not depend on the choice of the sets O_i.

Our first two theorems give criteria for the solvability of the problem (1.1) in $\mathbb{R}^N_+$.

Theorem 1.2 *Let $q > 1$ and $\sigma \in \mathfrak{M}^+_b(\mathbb{R}^{N-1})$. Then, the following statements are equivalent*

1 *There exists $C > 0$ such that the inequality*

$$\sigma(K) \leq C\, \mathrm{Cap}_{I_{\frac{2}{q}},q'}(K) \tag{1.9}$$

holds for any compact set $K \subset \mathbb{R}^{N-1}$.

2 *There exists $C > 0$ such that the relation*

$$\mathbf{G}\left[(\mathbf{P}[\sigma])^q\right] \leq C\mathbf{P}[\sigma] < \infty \quad a.e \text{ in } \quad \mathbb{R}^N_+ \tag{1.10}$$

holds.

3. *The problem*

$$\begin{aligned} -\Delta u &= u^q && \text{in } \mathbb{R}^N_+, \\ u &= \varepsilon\sigma && \text{in } \partial\mathbb{R}^N_+, \end{aligned} \tag{1.11}$$

has a positive solution for $\varepsilon > 0$ small enough.

Moreover, there is a constant $C_0 > 0$ such that if any one of the two statement **1** *and* **2** *holds with $C \le C_0$, then equation* (1.11) *admits a solution u with $\varepsilon = 1$ which satisfies*

$$u \asymp \mathbf{P}[\sigma]. \tag{1.12}$$

Conversely, if (1.11) *has a solution u with $\varepsilon = 1$, then the two statements* **1** *and* **2** *hold for some $C > 0$.*

As a consequence of Theorem 1.2 when $g(u, \nabla u) = |u|^{q-1}u$ ($q > 1$) and $\Omega = \mathbb{R}^N_+$, we prove that if (1.3) has a nonnegative solution u with $\sigma \in \mathfrak{M}^+_b(\mathbb{R}^{N-1})$, then

$$\sigma(B_r^{'}(y')) \le C r^{N - \frac{q+1}{q-1}} \tag{1.13}$$

for any ball $B_r^{'}(y')$ in $\mathbb{R}^{N-1}$ where $C = C(q, N)$ and $q > \frac{N+1}{N-1}$; if $1 < q \le \frac{N+1}{N-1}$, then $\sigma \equiv 0$. Conversely, if $q > \frac{N+1}{N-1}$, $d\sigma = f dz$ for some $f \ge 0$ which satisfies

$$\int_{B_r^{'}(y')} f^{1+\varepsilon} dz \le C r^{N-1-\frac{2(\varepsilon+1)}{q-1}} \tag{1.14}$$

for some $\varepsilon > 0$, then there exists a constant $C_0 = C_0(N, q)$ such that (1.1) has a nonnegative solution if $C \le C_0$. The above inequality is an analogue of the classical Fefferman-Phong condition [6]. In particular, (1.14) holds if f belongs to the Lorentz space $L^{\frac{(N-1)(q-1)}{2},\infty}(\mathbb{R}^{N-1})$.

We give sufficient conditions for the existence of weak solutions to (1.1) when $g(u, \nabla u) = |u|^{q_1-1}u|\nabla u|^{q_2}$, $q_1, q_2 \ge 0$, $q_1 + q_2 > 1$ and $q_2 < 2$.

Theorem 1.3 *Let $q_1, q_2 \ge 0, q_1 + q_2 > 1, q_2 < 2$ and $\sigma \in \mathfrak{M}(\mathbb{R}^{N-1})$ such that $\mathbf{P}[|\sigma|] < \infty$ a.e. in $\mathbb{R}^{N-1}$. Assume that there exists $C > 0$ such that for any Borel set $K \subset \mathbb{R}^{N-1}$ there holds*

$$|\sigma|(K) \le C \operatorname{Cap}_{I_{\frac{2-q_2}{q_1+q_2}},(q_1+q_2)'}(K). \tag{1.15}$$

Then the problem

$$\begin{aligned} -\Delta u &= |u|^{q_1-1}u|\nabla u|^{q_2} && \text{in } \mathbb{R}^N_+, \\ u &= \varepsilon\sigma && \text{in } \partial\mathbb{R}^N_+, \end{aligned} \tag{1.16}$$

has a solution for $\varepsilon > 0$ small enough and it satisfies

$$|u| \lesssim \mathbf{P}[|\sigma|], \quad |\nabla u| \lesssim \rho^{-1}\mathbf{P}[|\sigma|]. \tag{1.17}$$

Remark 1.4 *In any case and in view of* (1.7)*, if $d\sigma = f dz$, $f \in L^{\frac{(N-1)(q_1+q_2-1)}{2-q_2},\infty}(\mathbb{R}^{N-1})$ and $(N-1)(q_1+q_2-1) > 2 - q_2$ then* (1.15) *holds for some $C > 0$ and the problem* (1.16) *has a solution for $\varepsilon > 0$ small enough. However, we can see that condition* (1.15) *implies $\mathbf{P}[|\sigma|] < \infty$ a.e, see Theorem 2.6.*

In a bounded domain Ω we obtain existence results analogous to Theorem 1.2 and 1.3 provided the capacities on $\partial\Omega$ set in Definition 1.1 are used instead of the Riesz capacities.

Theorem 1.5 *Let $q > 1$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2 boundary and $\sigma \in \mathfrak{M}^+(\partial\Omega)$. Then, the following statements are equivalent:*

1 *There exists $C > 0$ such that the inequality*

$$\sigma(K) \le C \operatorname{Cap}^{\partial\Omega}_{\frac{2}{q},q'}(K) \tag{1.18}$$

for any Borel set $K \subset \partial\Omega$.

2 *There exists $C > 0$ such that the inequality*

$$\mathbf{G}\left[(\mathbf{P}[\sigma])^q\right] \leq C\mathbf{P}[\sigma] < \infty \quad a.e \ in \quad \Omega, \tag{1.19}$$

holds.

3. *The problem*

$$\begin{aligned} -\Delta u &= u^q && in\ \Omega, \\ u &= \varepsilon\sigma && on\ \ \partial\Omega, \end{aligned} \tag{1.20}$$

admits a positive solution for $\varepsilon > 0$ small enough.

Moreover, there is a constant $C_0 > 0$ such that if any one of the two statements **1** *and* **2** *holds with $C \leq C_0$, then equation* (1.20) *has a solution u with $\varepsilon = 1$ which satisfies*

$$u \asymp \mathbf{P}[\sigma]. \tag{1.21}$$

Conversely, if (1.20) *has a solution u with $\varepsilon = 1$, the above two statements* **1** *and* **2** *hold for some $C > 0$.*

From (1.8), we see that if $\sigma \in \mathfrak{M}^+(\partial\Omega)$ and $1 < q < \frac{N+1}{N-1}$, then (1.18) holds for some constant $C > 0$. Hence, in this case, the problem (1.20) has a positive solution for $\varepsilon > 0$ small enough.

Theorem 1.6 *Let $q_1, q_2 \geq 0, q_1 + q_2 > 1, q_2 < 2$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2 boundary and $\sigma \in \mathfrak{M}(\partial\Omega)$. Assume that there exists $C > 0$ such that the inequality*

$$|\sigma|(K) \leq C\,\mathrm{Cap}^{\partial\Omega}_{\frac{2-q_2}{q_1+q_2},(q_1+q_2)'}(K) \tag{1.22}$$

holds for any Borel set $K \subset \partial\Omega$. Then the problem

$$\begin{aligned} -\Delta u &= |u|^{q_1-1}u|\nabla u|^{q_2} && in\ \Omega, \\ u &= \varepsilon\sigma && on\ \ \partial\Omega, \end{aligned} \tag{1.23}$$

has a solution for $\varepsilon > 0$ small enough which satisfies (1.17).

Remark 1.7 *A discussion about the optimality of this condition, as well as the one of Theorem 1.3, is conducted in Remark 3.1. We define the subcritical range by*

$$(N-1)q_1 + Nq_2 < N+1 \quad or\ equivalently \quad (N-1)(q_1+q_2-1) < 2 - q_2. \tag{1.24}$$

If we assume that we are in the subcritical case, then problem (1.23) *has a solution for any measure $\sigma \in \mathfrak{M}_b(\partial\Omega)$ and $\varepsilon > 0$ small enough.*

2 Integral equations

Let Ω be either $\mathbb{R}^{N-1} \times (0, \infty)$ or Ω a bounded domain in $\mathbb{R}^N$ with a C^2 boundary $\partial\Omega$. For $0 \leq \alpha \leq \beta < N$, we denote

$$\mathbf{N}_{\alpha,\beta}(x,y) = \frac{1}{|x-y|^{N-\beta}\max\{|x-y|, \rho(x), \rho(y)\}^\alpha} \qquad \forall (x,y) \in \overline{\Omega} \times \overline{\Omega}. \tag{2.1}$$

We set

$$\mathbf{N}_{\alpha,\beta}[\nu](x) = \int_{\overline{\Omega}} \mathbf{N}_{\alpha,\beta}(x,y)d\nu(y) \qquad \forall \nu \in \mathfrak{M}^+(\overline{\Omega}),$$

and denote $\mathbf{N}_{\alpha,\beta}[f] := \mathbf{N}_{\alpha,\beta}[fdx]$ if $f \in L^1_{loc}(\Omega),\ f \geq 0$.

In this section, we are interested in the solvability of the following integral equations

$$U = \mathbf{N}_{\alpha,\beta}\left[U^q(\rho(.))^{\alpha_0}\right] + \mathbf{N}_{\alpha,\beta}[\omega] \tag{2.2}$$

where $\alpha_0 \geq 0$ and $\omega \in \mathfrak{M}^+(\overline{\Omega})$.

We follow the deep ideas developed by Kalton and Verbitsky in [10] who analyzed a PDE problem under the form of an integral equation. They proved a certain number of properties of this integral equation which are crucial for our study and, for the sake of completeness, we recall them here. Let X be a metric space and $\nu \in \mathfrak{M}^+(X)$. Let $\mathbf{K}$ be a Borel positive kernel function $\mathbf{K} : X \times X \mapsto (0, \infty]$ such that $\mathbf{K}$ is symmetric and $\mathbf{K}^{-1}$ satisfies a quasi-metric inequality, i.e. there is a constant $C \geq 1$ such that for all $x, y, z \in X$ we have

$$\frac{1}{\mathbf{K}(x,y)} \leq C\left(\frac{1}{\mathbf{K}(x,z)} + \frac{1}{\mathbf{K}(z,y)}\right).$$

Under these conditions, we can define the quasi-metric d by

$$d(x,y) = \frac{1}{\mathbf{K}(x,y)},$$

and denote by $\mathbb{B}_r(x) = \{y \in X : d(x,y) < r\}$ the open d-ball of radius $r > 0$ and center x. Note that this set can be empty.

For $\omega \in \mathfrak{M}^+(X)$, we define the potentials $\mathbf{K}\omega$ and $\mathbf{K}^\nu f$ by

$$\mathbf{K}\omega(x) = \int_X \mathbf{K}(x,y)d\omega(y), \quad \mathbf{K}^\nu f(x) = \int_X \mathbf{K}(x,y)f(y)d\nu(y),$$

and for $q > 1$, the capacity $\mathrm{Cap}^\nu_{\mathbf{K},q'}$ in X by

$$\mathrm{Cap}^\nu_{\mathbf{K},q'}(E) = \inf\left\{\int_X g^{q'}d\nu : g \geq 0, \mathbf{K}^\nu g \geq \chi_E\right\},$$

for any Borel set $E \subset X$.

Theorem 2.1 ([10]) *Let $q > 1$ and $\nu, \omega \in \mathfrak{M}^+(X)$ such that*

$$\int_0^{2r} \frac{\nu(\mathbb{B}_s(x))}{s}\frac{ds}{s} \leq C\int_0^r \frac{\nu(\mathbb{B}_s(x))}{s}\frac{ds}{s}, \tag{2.3}$$

$$\sup_{y \in \mathbb{B}_r(x)} \int_0^r \frac{\nu(\mathbb{B}_s(y))}{s}\frac{ds}{s} \leq C\int_0^r \frac{\nu(\mathbb{B}_s(x))}{s}\frac{ds}{s}, \tag{2.4}$$

for any $r > 0, x \in X$, where $C > 0$ is a constant. Then the following statements are equivalent:

1 *The equation $u = \mathbf{K}^\nu u^q + \varepsilon\mathbf{K}\omega$ has a solution for some $\varepsilon > 0$.*

2 *The inequality*

$$\int_E (\mathbf{K}\omega_E)^q d\sigma \leq C\omega(E) \tag{2.5}$$

holds for any Borel set $E \subset X$ where $\omega_E = \chi_E\omega$.

3. *For any Borel set $E \subset X$, there holds*

$$\omega(E) \leq C\,\mathrm{Cap}^\nu_{\mathbf{K},q'}(E). \tag{2.6}$$

4. *The inequality*

$$\mathbf{K}^{\nu}\left(\mathbf{K}\omega\right)^{q} \leq C\mathbf{K}\omega < \infty \quad \nu-a.e. \tag{2.7}$$

holds.

We check below that $N_{\alpha,\beta}$ satisfies all assumptions of $\mathbf{K}$ in Theorem 2.1.

Lemma 2.2 *$\mathbf{N}_{\alpha,\beta}$ is symmetric and satisfies the quasi-metric inequality.*

Proof. Clearly, $\mathbf{N}_{\alpha,\beta}$ is symmetric. Now we check the quasi-metric inequality associated to $\mathbf{N}_{\alpha,\beta}$ and $X=\overline{\Omega}$. For any $x,z,y\in\overline{\Omega}$ such that $x\neq y\neq z$, we have

$$\begin{aligned}|x-y|^{N-\beta+\alpha} &\lesssim |x-z|^{N-\beta+\alpha}+|z-y|^{N-\beta+\alpha}\\ &\lesssim \frac{1}{\mathbf{N}_{\alpha,\beta}(x,z)}+\frac{1}{\mathbf{N}_{\alpha,\beta}(z,y)}.\end{aligned}$$

Since $|\rho(x)-\rho(y)|\leq|x-y|$, there holds

$$\begin{aligned}|x-y|^{N-\beta}(\rho(x))^{\alpha}+|x-y|^{N-\beta}(\rho(y))^{\alpha} &\lesssim |x-y|^{N-\beta}(\min\{\rho(x),\rho(y)\})^{\alpha}+|x-y|^{N-\beta+\alpha}\\ &\lesssim \left(|x-z|^{N-\beta}+|z-y|^{N-\beta}\right)(\min\{\rho(x),\rho(y)\})^{\alpha}+|x-z|^{N-\beta+\alpha}+|z-y|^{N-\beta+\alpha}\\ &\lesssim \left((\rho(x))^{\alpha}|x-z|^{N-\beta}+|x-z|^{N-\beta+\alpha}\right)+\left((\rho(y))^{\alpha}|z-y|^{N-\beta}+|z-y|^{N-\beta+\alpha}\right)\\ &\lesssim \frac{1}{\mathbf{N}_{\alpha,\beta}(x,z)}+\frac{1}{\mathbf{N}_{\alpha,\beta}(z,y)}.\end{aligned}$$

Thus,

$$\frac{1}{\mathbf{N}_{\alpha,\beta}(x,y)}\lesssim\frac{1}{\mathbf{N}_{\alpha,\beta}(x,z)}+\frac{1}{\mathbf{N}_{\alpha,\beta}(z,y)}.$$

∎

Next we give sufficient conditions for (2.3), (2.4) to hold, in view of the applications that we develop in Sections 3 and 4.

Lemma 2.3 *If $d\nu(x)=(\rho(x))^{\alpha_0}\chi_{\Omega}dx$ with $\alpha_0\geq0$, then* (2.3) *and* (2.4) *hold.*

Proof. It is easy to see that for any $x\in\overline{\Omega},\ s>0$

$$B_{2^{-\frac{\alpha+1}{N-\beta}}S}(x)\cap\overline{\Omega}\subset\mathbb{B}_s(x)\subset B_S(x)\cap\overline{\Omega}, \tag{2.8}$$

with $S=\min\{s^{\frac{1}{N-\beta+\alpha}},s^{\frac{1}{N-\beta}}(\rho(x))^{-\frac{\alpha}{N-\beta}}\}$ and $\mathbb{B}_s(x)=\overline{\Omega}$ when $s>2^{\frac{\alpha N}{N-\alpha}}(diam\,(\Omega))^N$.
We show that for any $0\leq s<8diam\,(\Omega)$, $x\in\overline{\Omega}$

$$\nu(B_s(x))\asymp(\max\{\rho(x),s\})^{\alpha_0}s^N. \tag{2.9}$$

Indeed, take $0\leq s<8diam\,(\Omega)$, $x\in\overline{\Omega}$. There exist $\varepsilon=\varepsilon(\Omega)\in(0,1)$ and $x_s\in\Omega$ such that $B_{\varepsilon s}(x_s)\subset B_s(x)\cap\Omega$ and $\rho(x_s)>\varepsilon s$.
(a) If $0\leq s\leq\frac{\rho(x)}{4}$, so for any $y\in B_s(x)$, $\rho(y)\asymp\rho(x)$. Thus we obtain (2.9) because

$$\nu(B_s(x))\asymp(\rho(x))^{\alpha_0}|B_s(x)\cap\Omega|\asymp(\rho(x))^{\alpha_0}s^N.$$

(b) If $s>\frac{\rho(x)}{4}$, since $\rho(y)\leq\rho(x)+|x-y|<5s$ for any $y\in B_s(x)$, there holds $\nu(B_s(x))\lesssim s^{N+\alpha_0}$ and we have the following dichotomy:
(b.1) either $s\leq4\rho(x)$, then

$$\nu(B_s(x))\gtrsim\nu(B_{\frac{\rho(x)}{4}}(x))\asymp(\rho(x))^{\alpha_0+N}\gtrsim s^{N+\alpha_0};$$

(b.2) or $s \geq 4\rho(x)$, we have for any $y \in B_{\varepsilon s/2}(x_s)$, $\rho(y) \geq -|y-x_s| + \rho(x_s) > \varepsilon s/2$. It follows

$$\nu(B_s(x)) \gtrsim \nu(B_{\varepsilon s/2}(x_s)) \gtrsim s^{N+\alpha_0}.$$

Therefore (2.9) holds.

Next, for any $0 \leq s < 2^{\frac{(\alpha+1)(N-\beta+\alpha)}{N-\beta}} (diam\,(\Omega))^{N-\beta+\alpha}$ and $x \in \overline{\Omega}$, we have

$$\begin{aligned}\nu(\mathbb{B}_s(x)) &\asymp (\max\{\rho(x), \min\{s^{\frac{1}{N-\beta+\alpha}}, s^{\frac{1}{N-\beta}}(\rho(x))^{-\frac{\alpha}{N-\beta}}\}\})^{\alpha_0} \\ &\quad \times \left(\min\{s^{\frac{1}{N-\beta+\alpha}}, s^{\frac{1}{N-\beta}}(\rho(x))^{-\frac{\alpha}{N-\beta}}\}\right)^N \\ &\asymp \begin{cases} s^{\frac{\alpha_0+N}{N-\beta+\alpha}} & \text{if } \rho(x) \leq s^{\frac{1}{N-\beta+\alpha}}, \\ (\rho(x))^{\alpha_0-\frac{\alpha N}{N-\beta}} s^{\frac{N}{N-\beta}} & \text{if } \rho(x) \geq s^{\frac{1}{N-\beta+\alpha}}, \end{cases}\end{aligned}$$

and $\nu(\mathbb{B}_s(x)) = \nu(\overline{\Omega}) \asymp (diam\,(\Omega))^{\alpha_0+N}$ if $s > 2^{\frac{(\alpha+1)(N-\beta+\alpha)}{N-\beta}} (diam\,(\Omega))^{N-\beta+\alpha}$. We get,

$$\int_0^r \frac{\nu(\mathbb{B}_s(x))}{s}\frac{ds}{s} \asymp \begin{cases} (diam\,(\Omega))^{\alpha_0+\beta-\alpha} & \text{if } r > (diam\,(\Omega))^{N-\beta+\alpha}, \\ r^{\frac{\alpha_0+\beta-\alpha}{N-\beta+\alpha}} & \text{if } r \in ((\rho(x))^{N-\beta+\alpha}, (diam\,(\Omega))^{N-\beta+\alpha}], \\ (\rho(x))^{\alpha_0-\frac{\alpha N}{N-\beta}} r^{\frac{\beta}{N-\beta}} & \text{if } r \in (0, (\rho(x))^{N-\beta+\alpha}]. \end{cases}$$

Therefore (2.3) holds. It remains to prove (2.4). For any $x \in \overline{\Omega}$ and $r > 0$, it is clear that if $r > \frac{1}{2}(\rho(x))^{N-\beta+\alpha}$ we have

$$\sup_{y \in \mathbb{B}_r(x)} \int_0^r \frac{\nu(\mathbb{B}_s(y))}{s}\frac{ds}{s} \lesssim \min\{r^{\frac{\alpha_0+\beta-\alpha}{N-\beta+\alpha}}, (diam\,(\Omega))^{\alpha_0+\beta-\alpha}\},$$

from which inequality we obtain

$$\sup_{y \in \mathbb{B}_r(x)} \int_0^r \frac{\nu(\mathbb{B}_s(y))}{s}\frac{ds}{s} \lesssim \int_0^r \frac{\nu(\mathbb{B}_s(x))}{s}\frac{ds}{s}.$$

If $0 < r \leq \frac{1}{2}(\rho(x))^{N-\beta+\alpha}$, we have $\mathbb{B}_r(x) \subset B_{r^{\frac{1}{N-\beta}}(\rho(x))^{-\frac{\alpha}{N-\beta}}}(x)$ and $\rho(x) \asymp \rho(y)$ for all $y \in B_{r^{\frac{1}{N-\beta}}(\rho(x))^{-\frac{\alpha}{N-\beta}}}(x)$, thus

$$\begin{aligned}\sup_{y \in \mathbb{B}_r(x)} \int_0^r \frac{\nu(\mathbb{B}_s(y))}{s}\frac{ds}{s} &\leq \sup_{|y-x| < r^{\frac{1}{N-\beta}}(\rho(x))^{-\frac{\alpha}{N-\beta}}} \int_0^r \frac{\nu(\mathbb{B}_s(y))}{s}\frac{ds}{s} \\ &\asymp \sup_{|y-x| < r^{\frac{1}{N-\beta}}(\rho(x))^{-\frac{\alpha}{N-\beta}}} (\rho(y))^{\alpha_0-\frac{\alpha N}{N-\beta}} r^{\frac{\beta}{N-\beta}} \\ &\asymp (\rho(x))^{\alpha_0-\frac{\alpha N}{N-\beta}} r^{\frac{\beta}{N-\beta}} \\ &\asymp \int_0^r \frac{\nu(\mathbb{B}_s(x))}{s}\frac{ds}{s}.\end{aligned}$$

Therefore, (2.4) holds. ∎

Remark 2.4 *Lemma 2.2 and 2.3 in the case $\alpha = \beta = 2$ and $\alpha_0 = q+1$ had already been proved by Kalton and Verbitsky in [10].*

Definition 2.5 *For $\alpha_0 \geq 0, 0 \leq \alpha \leq \beta < N$ and $s > 1$, we define $\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}$ by*

$$\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}(E) = \inf\left\{\int_{\overline{\Omega}} g^s(\rho(x))^{\alpha_0}dx : g \geq 0, \mathbf{N}_{\alpha,\beta}[g(\rho(.))^{\alpha_0}] \geq \chi_E\right\}$$

for any Borel set $E \subset \overline{\Omega}$.

Clearly, we have

$$\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}(E) = \inf\left\{\int_{\overline{\Omega}} g^s(\rho(x))^{-\alpha_0(s-1)}dx : g \geq 0, \mathbf{N}_{\alpha,\beta}[g] \geq \chi_E\right\}$$

for any Borel set $E \subset \overline{\Omega}$. Furthermore we have by [1, Theorem 2.5.1],

$$\left(\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}(E)\right)^{1/s} = \sup\left\{\omega(E) : \omega \in \mathfrak{M}^+_b(E), ||\mathbf{N}_{\alpha,\beta}[\omega]||_{L^{s'}(\Omega,(\rho(.))^{\alpha_0}dx)} \leq 1\right\} \tag{2.10}$$

for any compact set $E \subset \overline{\Omega}$, where s' is the conjugate exponent of s.

Thanks to Lemma 2.2 and 2.3 , we can apply Theorem 2.1 and we obtain:

Theorem 2.6 *Let $\omega \in \mathfrak{M}^+(\overline{\Omega})$, $\alpha_0 \geq 0, 0 \leq \alpha \leq \beta < N$ and $q > 1$. Then the following statements are equivalent:*

1 *The equation $u = \mathbf{N}_{\alpha,\beta}[u^q(\rho(.))^{\alpha_0}] + \varepsilon\mathbf{N}_{\alpha,\beta}[\omega]$ has a solution for $\varepsilon > 0$ small enough.*

2 *The inequality*

$$\int_{E\cap\Omega}(\mathbf{N}_{\alpha,\beta}[\omega_E])^q(\rho(x))^{\alpha_0}dx \leq C\omega(E) \tag{2.11}$$

holds for some $C > 0$ and any Borel set $E \subset \overline{\Omega}$, $\omega_E = \omega\chi_E$.

3. *The inequality*

$$\omega(K) \leq C\,\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},q'}(K) \tag{2.12}$$

holds for some $C > 0$ and any compact set $K \subset \overline{\Omega}$.

4. *The inequality*

$$\mathbf{N}_{\alpha,\beta}\left[(\mathbf{N}_{\alpha,\beta}[\omega])^q(\rho(.))^{\alpha_0}\right] \leq C\mathbf{N}_{\alpha,\beta}[\omega] < \infty \quad a.e \text{ in } \quad \Omega \tag{2.13}$$

holds for some $C > 0$.

To apply the previous theorem we need the following result.

Proposition 2.7 *Let $q > 1$, $\nu, \omega \in \mathfrak{M}^+(X)$. Suppose that $A_1, A_2, B_1, B_2 : X\times X \mapsto [0, +\infty)$ are Borel positive Kernel functions with $A_1 \asymp A_2, B_1 \asymp B_2$. Then, the following statements are equivalent:*

1 *The equation $u = A_1^\nu u^q + \varepsilon B_1\omega$ ν-a.e has a solution for $\varepsilon > 0$ small enough.*

2 *The equation $u = A_2^\nu u^q + \varepsilon B_2\omega$ $\nu-$a.e has a solution for $\varepsilon > 0$ small enough.*

3. *The problem $u \asymp A_1^\nu u^q + \varepsilon B_1\omega$ ν-a.e has a solution for $\varepsilon > 0$ small enough.*

4. *The equation $u \gtrsim A_1^\nu u^q + \varepsilon B_1\omega$ ν-a.e has a solution for $\varepsilon > 0$ small enough.*

Proof. We prove only that 4 implies 2. Suppose that there exist $c_1 > 0, \varepsilon_0 > 0$ and a position Borel function u such that

$$A_1^\nu u^q + \varepsilon_0 B_1\omega \leq c_1 u.$$

Taken $c_2 > 0$ with $A_2 \leq c_2 A_1, B_2 \leq c_2 B$. We consider $u_{n+1} = A_2^\nu u_n^q + \varepsilon_0(c_1c_2)^{-\frac{q}{q-1}}B_2\omega$ and $u_0 = 0$ for any $n \geq 0$. Clearly, $u_n \leq (c_1c_2)^{-\frac{1}{q-1}}u$ for any n and $\{u_n\}$ is nondecreasing. Thus, $U = \lim_{n\to\infty} u_n$ is a solution of $U = A_2^\nu U^q + \varepsilon_0(c_1c_2)^{-\frac{q}{q-1}}B_2\omega$. ■

The following results provide some relations between the capacities $\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}$ and the Riesz capacities on $\mathbb{R}^{N-1}$ which allow to define the capacities on $\partial\Omega$.

Proposition 2.8 *Assume that* $\Omega = \mathbb{R}^{N-1} \times (0, \infty)$ *and let* $\alpha_0 \geq 0$ *such that*

$$-1 + s'(1 + \alpha - \beta) < \alpha_0 < -1 + s'(N - \beta + \alpha).$$

There holds

$$\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}(K \times \{0\}) \asymp \mathrm{Cap}_{I_{\beta-\alpha+\frac{\alpha_0+1}{s'}-1},s'}(K) \tag{2.14}$$

for any compact set $K \subset \mathbb{R}^{N-1}$,

Proof. The proof relies on an idea of [18, Corollary 4.20]. Thanks to [1, Theorem 2.5.1] and (2.10), we get (2.14) from the following estimate: for any $\mu \in \mathfrak{M}^+_b(\mathbb{R}^{N-1})$

$$||\mathbf{N}_{\alpha,\beta}[\mu \otimes \delta_{\{x_N=0\}}]||_{L^{s'}(\Omega,(\rho(.)))^{\alpha_0}dx)} \asymp ||I_{\beta-\alpha+\frac{\alpha_0+1}{s'}-1}[\mu]||_{L^{s'}(\mathbb{R}^{N-1})}, \tag{2.15}$$

where $I_\gamma[\mu]$ is the Riesz potential of μ in $\mathbb{R}^{N-1}$, i.e

$$I_\gamma[\mu](y) = \int_0^\infty \frac{\mu(B'_r(y))}{r^{N-1-\gamma}} \frac{dr}{r} \quad \forall\, y \in \mathbb{R}^{N-1},$$

with $B'_r(y)$ being a ball in $\mathbb{R}^{N-1}$. We have

$$\begin{aligned}
||\mathbf{N}_{\alpha,\beta}[\mu \otimes \delta_{\{x_N=0\}}]||^{s'}_{L^{s'}(\Omega,(\rho(.))^{\alpha_0}dx)} &= \int_{\mathbb{R}^{N-1}} \int_0^\infty \left(\int_{\mathbb{R}^{N-1}} \frac{d\mu(z)}{(|x'-z|^2 + x_N^2)^{\frac{N-\beta+\alpha}{2}}} \right)^{s'} x_N^{\alpha_0}\, dx_N dx' \\
&\asymp \int_{\mathbb{R}^{N-1}} \int_0^\infty \left(\int_{x_N}^\infty \frac{\mu(B'_r(x'))}{r^{N-\beta+\alpha}} \frac{dr}{r} \right)^{s'} x_N^{\alpha_0}\, dx_N dx'.
\end{aligned}$$

Notice that

$$\begin{aligned}
\int_0^\infty \left(\int_{x_N}^\infty \frac{\mu(B'_r(x'))}{r^{N-\beta+\alpha}} \frac{dr}{r} \right)^{s'} x_N^{\alpha_0}\, dx_N &\geq \int_0^\infty \left(\int_{x_N}^{2x_N} \frac{\mu(B'_r(x'))}{r^{N-\beta+\alpha}} \frac{dr}{r} \right)^{s'} x_N^{\alpha_0}\, dx_N \\
&\gtrsim \int_0^\infty \left(\frac{\mu(B'_{x_N}(x'))}{x_N^{N-\beta+\alpha-\frac{\alpha_0+1}{s'}}} \right)^{s'} \frac{dx_N}{x_N}.
\end{aligned}$$

On the other hand, using Hölder's inequality and Fubini's Theorem, we obtain

$$\begin{aligned}
\int_0^\infty \left(\int_{x_N}^\infty \frac{\mu(B'_r(x'))}{r^{N-\beta+\alpha}} \frac{dr}{r} \right)^{s'} x_N^{\alpha_0}\, dx_N &\leq \int_0^\infty \left(\int_{x_N}^\infty r^{-\frac{s}{2s'}} \frac{dr}{r} \right)^{\frac{s'}{s}} \int_{x_N}^\infty \left(\frac{\mu(B'_r(x'))}{r^{N-\beta+\alpha-\frac{1}{2s'}}} \right)^{s'} \frac{dr}{r} x_N^{\alpha_0}\, dx_N \\
&= C \int_0^\infty \int_{x_N}^\infty \left(\frac{\mu(B'_r(x'))}{r^{N-\beta+\alpha-\frac{1}{2s'}}} \right)^{s'} \frac{dr}{r} x_N^{\alpha_0-\frac{1}{2}}\, dx_N \\
&= C \int_0^\infty \int_0^r x_N^{\alpha_0-\frac{1}{2}}\, dx_N \left(\frac{\mu(B'_r(x'))}{r^{N-\beta+\alpha-\frac{1}{2s'}}} \right)^{s'} \frac{dr}{r} \\
&= C \int_0^\infty \left(\frac{\mu(B'_r(x'))}{r^{N-\beta+\alpha-\frac{\alpha_0+1}{s'}}} \right)^{s'} \frac{dr}{r}.
\end{aligned}$$

Thus,

$$||\mathbf{N}_{\alpha,\beta}[\mu \otimes \delta_{\{x_N=0\}}]||_{L^{s'}(\Omega,(\rho(.)))^{\alpha_0}dx)} \asymp \left(\int_{\mathbb{R}^{N-1}} \int_0^\infty \left(\frac{\mu(B'_r(y))}{r^{N-\beta+\alpha-\frac{\alpha_0+1}{s'}}} \right)^{s'} \frac{dr}{r} dy \right)^{1/s'}. \tag{2.16}$$

It implies (2.15) from [4, Theorem 2.3]. ∎

Proposition 2.9 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain a C^2 boundary. Assume $\alpha_0 \geq 0$ and $-1+s'(1+\alpha-\beta) < \alpha_0 < -1+s'(N-\beta+\alpha)$. Then there holds*

$$\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}(E) \asymp \mathrm{Cap}^{\partial\Omega}_{\beta-\alpha+\frac{\alpha_0+1}{s'}-1,s}(E) \tag{2.17}$$

for any compact set $E \subset \partial\Omega \subset \mathbb{R}^N$.

Proof. Let $K_1, ..., K_m$ be as in definition 1.1. We have

$$\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}(E) \asymp \sum_{i=1}^{m} \mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}(E \cap K_i),$$

for any compact set $E \subset \partial\Omega$. By definition 1.1, we need to prove that

$$\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}(E \cap K_i) \asymp \mathrm{Cap}_{G_{\beta-\alpha+\frac{\alpha_0+1}{s'}-1},s}(\tilde{T}_i(E \cap K_i)) \ \ \forall\ i = 1, 2, ..., m. \tag{2.18}$$

We can show that for any $\omega \in \mathfrak{M}^+_b(\partial\Omega)$ and $i = 1, ..., m$, there exists $\omega_i \in \mathfrak{M}^+_b(\tilde{T}_i(K_i))$ with $T_i(K_i) = \tilde{T}_i(K_i) \times \{x_N = 0\}$ such that

$$\omega_i(O) = \omega(T_i^{-1}(O \times \{0\}))$$

for all Borel set $O \subset \tilde{T}_i(K_i)$, its proof can be found in [1, Proof of Lemma 5.2.2]. Thanks to [1, Theorem 2.5.1], it is enough to show that for any $i \in \{1, 2, ..., m\}$ there holds

$$||\mathbf{N}_{\alpha,\beta}[\chi_{K_i}\omega]||_{L^{s'}(\Omega,(\rho(.)))^{\alpha_0}dx)} \asymp ||G_{\beta-\alpha+\frac{\alpha_0+1}{s'}-1}[\omega_i]||_{L^{s'}(\mathbb{R}^{N-1})}, \tag{2.19}$$

where $G_\gamma[\omega_i]$ $(0 < \gamma < N-1)$ is the Bessel potential of ω_i in $\mathbb{R}^{N-1}$, i.e

$$G_\gamma[\omega_i](x) = \int_{\mathbb{R}^{N-1}} G_\gamma(x-y)d\omega_i(y).$$

Indeed, we have

$$\begin{aligned}
&||\mathbf{N}_{\alpha,\beta}[\omega\chi_{K_i}]||^{s'}_{L^{s'}(\Omega,(\rho(.)))^{\alpha_0}dx)} = \int_\Omega \left(\int_{K_i} \frac{d\omega(z)}{|x-z|^{N-\beta+\alpha}}\right)^{s'} (\rho(x))^{\alpha_0}dx \\
&= \int_{O_i\cap\Omega} \left(\int_{K_i} \frac{d\omega(z)}{|x-z|^{N-\beta+\alpha}}\right)^{s'} (\rho(x))^{\alpha_0}dx + \int_{\Omega\backslash O_i} \left(\int_{K_i} \frac{d\omega(z)}{|x-z|^{N-\beta+\alpha}}\right)^{s'} (\rho(x))^{\alpha_0}dx \\
&\asymp \int_{O_i\cap\Omega} \left(\int_{K_i} \frac{d\omega(z)}{|x-z|^{N-\beta+\alpha}}\right)^{s'} (\rho(x))^{\alpha_0}dx + (\omega(K_i))^{s'}.
\end{aligned}$$

Here we used $|x-z| \asymp 1$ for any $x \in \Omega\backslash O_i, z \in K_i$.
By a standard change of variable we obtain

$$\begin{aligned}
&\int_{O_i\cap\Omega} \left(\int_{K_i} \frac{d\omega(z)}{|x-z|^{N-\beta+\alpha}}\right)^{s'} (\rho(x))^{\alpha_0}dx + (\omega(K_i))^{s'} \\
&= \int_{T_i(O_i\cap\Omega)} \left(\int_{K_i} \frac{d\omega(z)}{|T_i^{-1}(y)-z|^{N-\beta+\alpha}}\right)^{s'} (\rho(T_i^{-1}(y)))^{\alpha_0}|\mathbf{J}_{T_i}(T_i^{-1}(y))|^{-1}dy + (\omega(K_i))^{s'} \\
&\asymp \int_{B_1(0)\cap\{x_N>0\}} \left(\int_{K_i} \frac{d\omega(z)}{|y-T_i(z)|^{N-\beta+\alpha}}\right)^{s'} y_N^{\alpha_0}dy + (\omega(K_i))^{s'} \quad \text{with } y = (y', y_N),
\end{aligned}$$

since $|T_i^{-1}(y) - z| \asymp |y - T_i(z)|$, $|\mathbf{J}_{T_i}(T_i^{-1}(y))| \asymp 1$ and $\rho(T_i^{-1}(y)) \asymp y_N$ for all $(y,z) \in T_i(O_i \cap \Omega) \times K_i$. From the definition of ω_i, we have

$$\begin{aligned}
&\int_{B_1(0)\cap\{x_N>0\}} \left(\int_{K_i} \frac{1}{|y - T_i(z)|^{N-\beta+\alpha}} d\omega(z) \right)^{s'} y_n^{\alpha_0} dy + (\omega(K_i))^{s'} \\
&= \int_{B_1(0)\cap\{x_N>0\}} \left(\int_{\tilde{T}_i(K_i)} \frac{1}{(|y' - \xi|^2 + y_N^2)^{\frac{N-\beta+\alpha}{2}}} d\omega_i(\xi) \right)^{s'} y_N^{\alpha_0} dy_N dy' + (\omega(K_i))^{s'} \\
&\asymp \int_{\mathbb{R}^{N-1}} \int_0^\infty \left(\int_{\min\{y_N,R\}}^{2R} \frac{\omega_i(B'_r(y'))}{r^{N-\beta+\alpha}} \frac{dr}{r} \right)^{s'} y_N^{\alpha_0} dy_N dy' \quad \text{with} \quad R = \operatorname{diam}(\Omega).
\end{aligned}$$

As in the proof of Proposition 2.8, there holds

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}} \int_0^\infty \left(\int_{\min\{y_N,R\}}^{2R} \frac{\omega_i(B'_r(y'))}{r^{N-\beta+\alpha}} \frac{dr}{r} \right)^{s'} y_N^{\alpha_0} dy_N dy' \\
&\asymp \int_{\mathbb{R}^{N-1}} \int_0^{2R} \left(\frac{\omega_i(B'_r(y'))}{r^{N-\beta+\alpha-\frac{\alpha_0+1}{s'}}} \right)^{s'} \frac{dr}{r} dy'.
\end{aligned}$$

Therefore, we get (2.19) from [4, Theorem 2.3]. ∎

Remark 2.10 *Proposition 2.8 and 2.9 with $\alpha = \beta = 2, \alpha_0 = q+1$ were demonstrated by Verbitsky in [5, Apppendix B], using an alternative approach.*

3 Proof of the main results

We denote

$$\mathbf{P}[\sigma](x) = \int_{\partial\Omega} \mathrm{P}(x,z) d\sigma(z), \quad \mathbf{G}[f](x) = \int_\Omega \mathrm{G}(x,y) f(y) dy$$

for any $\sigma \in \mathfrak{M}(\partial\Omega), f \in L^1_\rho(\Omega), f \geq 0$. Then the unique weak solution of

$$\begin{aligned}
-\Delta u &= f \qquad &&\text{in } \Omega, \\
u &= \sigma &&\text{on } \partial\Omega,
\end{aligned}$$

can be represented by

$$u(x) = \mathbf{G}[f](x) + \mathbf{P}[\sigma](x) \quad \forall\, x \in \Omega.$$

We recall below some classical estimates for the Green and the Poisson kernels.

$$\begin{aligned}
\mathrm{G}(x,y) &\asymp \min\left\{ \frac{1}{|x-y|^{N-2}}, \frac{\rho(x)\rho(y)}{|x-y|^N} \right\}, \\
\mathrm{P}(x,z) &\asymp \frac{\rho(x)}{|x-z|^N},
\end{aligned}$$

and

$$|\nabla_x \mathrm{G}(x,y)| \lesssim \frac{\rho(y)}{|x-y|^N} \min\left\{ 1, \frac{|x-y|}{\sqrt{\rho(x)\rho(y)}} \right\}, \quad |\nabla_x \mathrm{P}(x,z)| \lesssim \frac{1}{|x-z|^N},$$

for any $(x,y,z) \in \Omega \times \Omega \times \partial\Omega$, see [2]. Since $|\rho(x) - \rho(y)| \leq |x-y|$ we have

$$\max\left\{ \rho(x)\rho(y), |x-y|^2 \right\} \asymp \max\left\{ |x-y|, \rho(x), \rho(y) \right\}^2.$$

Thus,

$$\min\left\{1,\left(\frac{|x-y|}{\sqrt{\rho(x)\rho(y)}}\right)^{\gamma}\right\}\asymp\frac{|x-y|^{\gamma}}{\left(\max\left\{|x-y|,\rho(x),\rho(y)\right\}\right)^{\gamma}}\quad\text{for}\quad\gamma>0. \tag{3.1}$$

Therefore,

$$\mathrm{G}(x,y)\asymp\rho(x)\rho(y)\mathbf{N}_{2,2}(x,y),\quad\mathrm{P}(x,z)\asymp\rho(x)\mathbf{N}_{\alpha,\alpha}(x,z) \tag{3.2}$$

and

$$|\nabla_x\mathrm{G}(x,y)|\lesssim\rho(y)\mathbf{N}_{1,1}(x,y),\quad|\nabla_x\mathrm{P}(x,z)|\lesssim\mathbf{N}_{\alpha,\alpha}(x,z) \tag{3.3}$$

for all $(x,y,z)\in\overline{\Omega}\times\overline{\Omega}\times\partial\Omega$, $\alpha\geq 0$.

Proof of Theorem 1.2 and Theorem 1.5. By (3.2), the following equivalence holds

$$\mathbf{G}\left[(\mathbf{P}[\sigma])^q\right]\lesssim\mathbf{P}[\sigma]<\infty\ \ a.e\text{ in }\ \Omega.\Longleftrightarrow\quad\mathbf{N}_{2,2}\left[(\mathbf{N}_{2,2}[\sigma])^q\rho^{q+1}\right]\lesssim\mathbf{N}_{2,2}[\sigma]<\infty\ \ a.e\text{ in }\ \Omega.$$

Furthermore

$$U\asymp\mathbf{G}[U^q]+\mathbf{P}[\sigma]\Longleftrightarrow U\asymp\rho\mathbf{N}_{2,2}[\rho U^q]+\rho\mathbf{N}_{2,2}[\sigma],$$

which in turn is equivalent to

$$V\asymp\mathbf{N}_{2,2}[\rho^{q+1}V^q]+\mathbf{N}_{2,2}[\sigma]\text{ with }V=U\rho^{-1}.$$

By Proposition 2.8 and 2.9 we have:

$$\mathrm{Cap}_{I_{\frac{2}{q}},q'}(K)\asymp\mathrm{Cap}^{q+1}_{\mathbf{N}_{2,2},q'}(K\times\{0\})\qquad\forall\,K\subset\mathbb{R}^{N-1},K\text{ compact}.$$

if $\Omega=\mathbb{R}^N_+$, and

$$\mathrm{Cap}^{\partial\Omega}_{\frac{2}{q},q'}(K)\asymp\mathrm{Cap}^{q+1}_{\mathbf{N}_{2,2},q'}(K)\qquad\forall\,K\subset\partial\Omega,K\text{ compact}.$$

if Ω is a bounded domain. Thanks to Theorem (2.6) with $\omega=\sigma$, $\alpha=2,\beta=2,\alpha_0=q+1$ and proposition 2.7, we get the results.

∎

Proof of Theorem 1.3 and 1.6. By (3.2) and (3.3), we have

$$\mathrm{G}(x,y)\leq C\rho(x)\rho(y)\mathbf{N}_{1,1}(x,y),\quad|\nabla_x\mathrm{G}(x,y)|\leq C\rho(y)\mathbf{N}_{1,1}(x,y), \tag{3.4}$$
$$\mathrm{P}(x,z)\leq C\rho(y)\mathbf{N}_{1,1}(x,z),\quad|\nabla_x\mathrm{P}(x,z)|\leq C\mathbf{N}_{1,1}(x,z), \tag{3.5}$$

for all $(x,y,z)\in\Omega\times\Omega\times\partial\Omega$ and for some constant $C>0$.
For any $u\in W^{1,1}_{loc}(\Omega)$, we set

$$\mathbf{F}(u)(x)=\int_{\Omega}\mathrm{G}(x,y)|u(y)|^{q_1-1}u(y)|\nabla u(y)|^{q_2}dy+\int_{\partial\Omega}\mathrm{P}(x,z)d\sigma(z).$$

Using (3.4) and (3.5), we have

$$|\mathbf{F}(u)|\leq C\rho(.)\mathbf{N}_{1,1}\left[|u|^{q_1}|\nabla u|^{q_2}\rho(.)\right]+C\rho(.)\mathbf{N}_{1,1}[|\sigma|],$$
$$|\nabla\mathbf{F}(u)|\leq C\mathbf{N}_{1,1}\left[|u|^{q_1}|\nabla u|^{q_2}\rho(.)\right]+C\mathbf{N}_{1,1}[|\sigma|].$$

Therefore, we can easily see that if

$$\mathbf{N}_{1,1}\left[(\mathbf{N}_{1,1}[|\sigma|])^{q_1+q_2}(\rho(.))^{q_1+1}\right]\leq\frac{(q_1+q_2-1)^{q_1+q_2-1}}{(C(q_1+q_2))^{q_1+q_2}}\mathbf{N}_{1,1}[|\sigma|]<\infty\ \ a.e\text{ in }\quad\Omega \tag{3.6}$$

holds, then $\mathbf{F}$ is the map from $\mathbf{E}$ to $\mathbf{E}$, where

$$\mathbf{E}=\left\{u\in W^{1,1}_{loc}(\Omega):|u|\leq\lambda\rho(.)\mathbf{N}_{1,1}[|\sigma|],\ |\nabla u|\leq\lambda\mathbf{N}_{1,1}[|\sigma|]\quad\text{a.e in }\ \Omega\right\}$$

with $\lambda=\frac{C(q_1+q_2)}{q_1+q_2-1}$.
Assume that (3.6) holds. We denote $\mathcal{S}$ by the subspace of functions $f\in W^{1,1}_{loc}(\Omega)$ with norm

$$||f||_{\mathcal{S}}=||f||_{L^{q_1+q_2}(\Omega,(\rho(.))^{1-q_2}dx)}+|||\nabla f|||_{L^{q_1+q_2}(\Omega,(\rho(.))^{1+q_2}dx)}<\infty.$$

Clearly, $\mathbf{E}\subset\mathcal{S}$, $\mathbf{E}$ is closed under the strong topology of $\mathcal{S}$ and convex.
On the other hand, it is not difficult to show that $\mathbf{F}$ is continuous and $\mathbf{F}(\mathbf{E})$ is precompact in $\mathcal{S}$. Consequently, by Schauder's fixed point theorem, there exists $u\in\mathbf{E}$ such that $\mathbf{F}(u)=u$. Hence, u is a solution of (1.16)-(1.23) and it satisfies

$$|u|\leq\lambda\rho(.)\mathbf{N}_{1,1}[|\sigma|],\ |\nabla u|\leq\lambda\mathbf{N}_{1,1}[|\sigma].$$

Thanks to Theorem 2.6 and Proposition 2.8, 2.9, we verify that assumptions (1.15) and (1.23) in Theorem 1.3 and 1.6 are equivalent to (3.6). This completes the proof of the Theorems. ■

Remark 3.1 *We do not know whether conditions* (1.15) *and* (1.22) *are optimal or not. It is noticeable that if* $\mathbf{P}[|\sigma|]\in L^{q_1+q_2}(\Omega,\rho^{1-q_2}dx)$, *it is proved in [14, Th 1.1] that, if* Ω *is a ball, then* $|\sigma|$ *belongs to the Besov-Sobolev space* $B^{-\frac{2-q_2}{q_1+q_2},q_1+q_2}(\partial\Omega)$. *Therefore inequality*

$$|\sigma|(K)\leq C\left(\mathrm{Cap}^{\partial\Omega}_{\frac{2-q_2}{q_1+q_2},(q_1+q_2)'}(K)\right)^{\frac{1}{(q_1+q_2)'}}$$

holds for any Borel set $K\subset\partial\Omega$, *and it is a necessary condition for* (1.22) *to hold since* $\frac{1}{(q_1+q_2)'}<1$. *In a general* C^2 *bounded domain, it is easy to see that this property, proved in a particular case in [13, Th 2.2] is still valid thanks to the equivalence relation (2.23) therein between Poisson's kernels, see also the proof of Proposition 2.9. The difficulty for obtaining a necessary condition of existence lies in the fact that, if the inequality* $u\geq\mathbf{P}[\sigma]$ *is clear,* $|\nabla u|\gtrsim\rho^{-1}\mathbf{P}[\sigma]$ *is not true. It can also be shown that if*

$$|u|^{q_1}|\nabla u|^{q_2}\leq C(\mathbf{G}(|\sigma|))^{q_1}(\rho\mathbf{N}_{1,1}[|\sigma|])^{q_2}\in L^1(\Omega,\rho(.)dx),$$

then σ *is absolutely continuous with respect to* $\mathrm{Cap}^{\partial\Omega}_{\frac{2-q_2}{q_1+q_2},(q_1+q_2)'}$.

4 Extension to Schrödinger operators with Hardy potentials

We can apply Theorem 2.6 to solve the problem

$$\begin{aligned}-\Delta u-\tfrac{\kappa}{\rho^2}u&=u^q &&\text{in }\Omega,\\ u&=\sigma &&\text{on }\ \partial\Omega,\end{aligned}$$

where $\kappa\in[0,\frac14]$ and $\sigma\in\mathfrak{M}^+(\partial\Omega)$.

Let $\mathrm{G}_\kappa,\mathrm{P}_\kappa$ be the Green kernel and Poisson kernel of $-\Delta-\frac{\kappa}{\rho^2}$ in Ω with $\kappa\in[0,\frac14]$. It is proved that

$$\mathrm{G}_\kappa(x,y)\asymp\min\left\{\frac{1}{|x-y|^{N-2}},\frac{(\rho(x)\rho(y))^{\frac{1+\sqrt{1-4\kappa}}{2}}}{|x-y|^{N-1+\sqrt{1-4\kappa}}}\right\},$$
$$\mathrm{P}_\kappa(x,z)\asymp\frac{(\rho(x))^{\frac{1+\sqrt{1-4\kappa}}{2}}}{|x-z|^{N-1+\sqrt{1-4\kappa}}},$$

for all $(x,y,z)\in\overline{\Omega}\times\overline{\Omega}\times\partial\Omega$, see [7, 12, 8]. Therefore, from (3.1) we get

$$\mathrm{G}_\kappa(x,y)\asymp(\rho(x)\rho(y))^{\frac{1+\sqrt{1-4\kappa}}{2}}\mathbf{N}_{1+\sqrt{1-4\kappa},2}(x,y), \tag{4.1}$$

$$\mathrm{P}_\kappa(x,z)\asymp(\rho(x))^{\frac{1+\sqrt{1-4\kappa}}{2}}\mathbf{N}_{\alpha,1-\sqrt{1-4\kappa}+\alpha}(x,z), \tag{4.2}$$

for all $(x,y,z)\in\overline{\Omega}\times\overline{\Omega}\times\partial\Omega$, $\alpha\geq 0$. We denote

$$\mathbf{P}_\kappa[\sigma](x)=\int_{\partial\Omega}\mathrm{P}_\kappa(x,z)d\sigma(z),\quad \mathbf{G}_\kappa[f](x)=\int_\Omega \mathrm{G}_\kappa(x,y)f(y)dy$$

for any $\sigma\in\mathfrak{M}^+(\partial\Omega)$, $f\in L^1(\Omega,\rho^{\frac{1+\sqrt{1-4\kappa}}{2}}dx)$, $f\geq 0$. Then the unique weak solution of

$$\begin{aligned} -\Delta u-\tfrac{\kappa}{\rho^2}u&=f \qquad &&\text{in } \Omega,\\ u&=\sigma &&\text{on } \partial\Omega, \end{aligned}$$

satisfies the following integral equation [8]

$$u=\mathbf{G}_\kappa[f]+\mathbf{P}_\kappa[\sigma]\quad\text{a.e. in }\Omega.$$

As in the proofs of Theorem 1.2 and Theorem 1.5 the relation

$$\mathbf{G}_\kappa\left[(\mathbf{P}_\kappa[\sigma])^q\right]\lesssim\mathbf{P}_\kappa[\sigma]<\infty\quad\text{a.e in}\quad\Omega,$$

is equivalent to

$$\mathbf{N}_{1+\sqrt{1-4\kappa},2}\left[\left(\mathbf{N}_{1+\sqrt{1-4\kappa},2}[\sigma]\right)^q\rho^{\frac{(q+1)(1+\sqrt{1-4\kappa})}{2}}\right]\lesssim\mathbf{N}_{1+\sqrt{1-4\kappa},2}[\sigma]<\infty\quad\text{a.e in}\quad\Omega,$$

and the relation

$$U\asymp\mathbf{G}_\kappa[U^q]+\mathbf{P}_\kappa[\sigma],$$

is equivalent to

$$V\asymp\mathbf{N}_{1+\sqrt{1-4\kappa},2}[\rho^{\frac{(q+1)(1+\sqrt{1-4\kappa})}{2}}V^q]+\mathbf{N}_{1+\sqrt{1-4\kappa},2}[\sigma]\quad\text{with } V=U\rho^{-\frac{1+\sqrt{1-4\kappa}}{2}}.$$

Thanks to Theorem 2.6 with $\omega=\sigma$, $\alpha=1+\sqrt{1-4\kappa},\beta=2,\alpha_0=\frac{(q+1)(1+\sqrt{1-4\kappa})}{2}$ and proposition 2.7, 2.8, 2.9, we obtain.

Theorem 4.1 *Let $q>1, 0\leq\kappa\leq\frac{1}{4}$ and $\sigma\in\mathfrak{M}^+(\partial\Omega)$. Then, the following statements are equivalent*

1 *There exists $C>0$ such that the following inequalities hold*

$$\sigma(O)\leq C\,\mathrm{Cap}_{I_{\frac{q+3-(q-1)\sqrt{1-4\kappa}}{2q}},q'}(O) \tag{4.3}$$

for any Borel set $O\subset\mathbb{R}^{N-1}$ if $\Omega=\mathbb{R}^N_+$ and

$$\sigma(O)\leq C\,\mathrm{Cap}^{\partial\Omega}_{\frac{q+3-(q-1)\sqrt{1-4\kappa}}{2q},q'}(O) \tag{4.4}$$

for any Borel set $O\subset\partial\Omega$ if Ω is a bounded domain.

2 *There exists $C>0$ such that the inequality*

$$\mathbf{G}_\kappa\left[(\mathbf{P}_\kappa[\sigma])^q\right]\leq C\mathbf{P}_\kappa[\sigma]<\infty\quad a.e\ in\quad\Omega, \tag{4.5}$$

holds.

3. *Problem*

$$\begin{array}{rll} -\Delta u - \frac{\kappa}{\rho^2} u = u^q & \quad \textit{in } \Omega, \\ u = \varepsilon\sigma & \quad \textit{on } \partial\Omega, \end{array} \tag{4.6}$$

has a positive solution for $\varepsilon > 0$ small enough.

Moreover, there is a constant $C_0 > 0$ such that if any one of the two statements **1** *and* **2** *holds with $C \leq C_0$, then equation 4.6 has a solution u with $\varepsilon = 1$ which satisfies*

$$u \asymp \mathbf{P}_\kappa[\sigma]. \tag{4.7}$$

Conversely, if (4.6) *has a solution u with $\varepsilon = 1$, then the two statements* **1** *and* **2** *hold for some $C > 0$.*

Remark 4.2 *The problem* (4.6) *admits a subcritical range*

$$1 < q < \frac{N + \frac{1+\sqrt{1-4\kappa}}{2}}{N + \frac{1+\sqrt{1-4\kappa}}{2} - 2}.$$

If the above inequality, the problem can be solved with any positive measure provided $\sigma(\partial\Omega)$ is small enough. The role of this critical exponent has been pointed out in [12] and [8] for the removability of boundary isolated singularities of solutions of

$$-\Delta u - \frac{\kappa}{\rho^2} u + u^q = 0 \textit{ in } \Omega$$

i.e. solutions which vanish on the boundary except at one point. Furthermore the complete study of the problem

$$\begin{array}{rll} -\Delta u - \frac{\kappa}{\rho^2} u + u^q = 0 & \quad \textit{in } \Omega, \\ u = \sigma & \quad \textit{on } \partial\Omega, \end{array} \tag{4.8}$$

is performed in [8] in the supercritical range

$$q \geq \frac{N + \frac{1+\sqrt{1-4\kappa}}{2}}{N + \frac{1+\sqrt{1-4\kappa}}{2} - 2}.$$

The necessary and sufficient condition is therein expressed in terms of the absolute continuity of σ with respect to the $\mathrm{Cap}_{I_{\frac{q+3-(q-1)\sqrt{1-4\kappa}}{2q}},q'}$*-capacity.*

References

[1] D. R. Adams, L.I. Heberg, *Function Spaces and Potential Theory*, Grundlehren der Mathematischen Wisenschaften **31**, Springer-Verlag (1999).

[2] M.F. Bidaut-Véron, L. Vivier, *An elliptic semilinear equation with source term involving boundary measures: the subcritical case* , Rev. Mat. Iberoamericana, **16** (2000), 477-513.

[3] M.F. Bidaut-Véron, C. Yarur, *Semilinear elliptic equations and systems with measure data: existence and a priori estimates*, Advances in Diff. Equ. 7 (2002), 257-296.

[4] M. F. Bidaut-Véron, H. Nguyen Quoc, L. Véron, *Quasilinear Lane-Emden equations with absorption and measure data*, J. Math. Pures Appl. **102**, 315-337 (2014).

[5] E.B. Dynkin, Superdiffusions and positive solutions of nonlinear partial differential equations, American Mathematical Society, Providence, RI, 2004.

[6] C. Fefferman, *The uncertainty principle*, Bull. Amer. Math.Soc. **9** (1983),129-206.

[7] S. Filippas, L. Moschini, A. Tertikas, *Sharp two-sided heat kernel estimates for critical Schrödinger operators on bounded domains*, Comm. Math. Phys. **273** (2007), 237-281.

[8] K. T. Gkikas, L. Véron, *Boundary singularities of solutions of semilinear elliptic equations with critical Hardy potentials*, Nonlinear Anal. T. M. & A., doi:10.1016/j.na.2015.03.004, to appear (2015).

[9] A. Gmira, L. Véron, *Boundary singularities of solutions of some nonlinear elliptic equations*, Duke Math. J. **64** (1991), 271-324.

[10] N.J. Kalton, I.E. Verbitsky, *Nonlinear equations and weighted norm inequality*, Trans. Amer. Math. Soc. **351** (1999) 3441-3497.

[11] M. Marcus, T. Nguyen Phuoc, *Positive solutions of quasilinear elliptic equations with subquadratic growth in the gradient*, arXiv:1311.7519v1 (2013).

[12] M. Marcus, P. T. Nguyen, *Moderate solutions of semilinear elliptic equations with Hardy potential*, arXiv:1407.3572v1 (2014).

[13] M. Marcus, L. Véron, *Removable singularities and boundary trace*, J. Math. Pures Appl. **80** (2001), 879-900.

[14] M. Marcus, L. Véron, *On a New Characterization of Besov Spaces with Negative Exponents*, Around the research of Vladimir Maz'ya. I, 273-284, Int. Math. Ser. (N. Y.) **11**, Springer, New York, 2010.

[15] M. Marcus, L. Véron, *Boundary trace of positive solutions of semilinear elliptic equations in Lipschitz domains: the subcritical case*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **10** (2011), 913-984.

[16] M. Marcus, L. Véron, *Nonlinear Second Order Elliptic Equations Involving Measures*, De Gruyter Series in Nonlinear Analysis and Applications **21**, De Gruyter, Berlin (2014), xiv+248 pp.

[17] T. Nguyen Phuoc, L. Véron, *Boundary singularities of solutions to elliptic viscous HamiltonJacobi equations*, J. Funct. An. **263** (2012) 1487-1538.

[18] Quoc-Hung Nguyen, *Potential estimates and quasilinear parabolic equations with measure data,*, hal-00989464, version 2 (2014), submitted.